Non-vanishing of Poincaré series

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Examples of Poincaré series on $\mathcal{H} = \mathbb{C}_{\Im(\tau)>0}$

For $k \in 4 + 2\mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}_{>0}$, writing $j\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau\right) = c\tau + d$:

- the classical holomorphic Eisenstein series $E_k \in M_k(\mathrm{SL}_2(\mathbb{Z}))$,

$$E_k(\tau) := \sum_{\gamma \in \mathrm{SL}_2(\mathbb{Z})_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} j(\gamma, \tau)^{-k}, \qquad \tau \in \mathcal{H},$$

  $\rightsquigarrow$ does not vanish at the cusps, so does not vanish identically

- the classical Poincaré series $\psi_{n,k} \in S_k(\mathrm{SL}_2(\mathbb{Z}))$,

$$\psi_{n,k}(\tau) := \sum_{\gamma \in \mathrm{SL}_2(\mathbb{Z})_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} e^{2\pi i n \gamma.\tau} j(\gamma, \tau)^{-k}, \qquad \tau \in \mathcal{H}$$

  $\rightsquigarrow$?

Which $\psi_{n,k}$ are identically zero?

For $k \in 4 + 2\mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}_{>0}$, $\psi_{n,k} \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ is defined by

$$\psi_{n,k}(\tau) := \sum_{\gamma \in \mathrm{SL}_2(\mathbb{Z})_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} e^{2\pi i n \gamma.\tau} j(\gamma, \tau)^{-k}, \qquad \tau \in \mathcal{H}.$$

- $d_k := \dim_{\mathbb{C}} S_k(\mathrm{SL}_2(\mathbb{Z})) = 0$ for $k \in \{4, 6, 8, 10, 14\}$.
- $\{\psi_{1,k} \ldots, \psi_{d_k,k}\}$ is a basis of $S_k(\mathrm{SL}_2(\mathbb{Z}))$.
- Ideas for $n > d_k$:

Rankin (1980)	estimating the n^{th} Fourier coefficient of $\psi_{n,k}$	$n \leq k^{2-\frac{C}{\log\log k}}$ for $k >> 0$
Rhoades (2011)	reformulation in terms of existence of weakly modular forms with a given principal part of Fourier expansion	$n \leq \frac{1}{12}(k-2)$
Muić (2011)	integral non-vanishing criterion	$n \leq \frac{1}{4\pi}\left(k - \frac{8}{3}\right)$

Poincaré series

Let:

- G be a locally compact Hausdorff group, second-countable and unimodular, with Haar measure dg
- $\Lambda \subseteq \Gamma$ be discrete subgroups of G
- $\chi : \Gamma \to \mathbb{C}^\times$ be a unitary character
- $\varphi : G \to \mathbb{C}$ be a measurable function such that:
  (F1) $\varphi(\lambda g) = \chi(\lambda)\varphi(g), \quad \lambda \in \Lambda,\ g \in G.$
  (F2) $|\varphi| \in L^1(\Lambda \backslash G)$.

Lemma

The Poincaré series

$$\left(P_{\Lambda\backslash\Gamma,\chi}\varphi\right)(g) := \sum_{\gamma\in\Lambda\backslash\Gamma} \overline{\chi(\gamma)}\,\varphi(\gamma g)$$

converges absolutely almost everywhere on G, and $\left|P_{\Lambda\backslash\Gamma,\chi}\varphi\right| \in L^1(\Gamma\backslash G)$.

Theorem 1 (Muić 2009; Ž. 2018)

We have

$$\int_{\Gamma\backslash G} \left|\left(P_{\Lambda\backslash\Gamma,\chi}\varphi\right)(g)\right| \, dg > 0$$

if there exists a Borel-measurable set $C \subseteq G$ such that:

(C1) $CC^{-1} \cap \Gamma \subseteq \Lambda$.

(C2) We have

$$\int_{\Lambda\backslash\Lambda C} |\varphi(g)| \, dg > \frac{1}{2} \int_{\Lambda\backslash G} |\varphi(g)| \, dg$$

for some measurable function $| \cdot | : \mathbb{C} \to \mathbb{R}_{\geq 0}$ such that:

(B1) $|0| = 0$.

(B2) $|z| = \big| \, |z| \, \big|, \quad z \in \mathbb{C}$.

(B3) $\left| \sum_{n=1}^{\infty} z_n \right| \leq \sum_{n=1}^{\infty} |z_n|$ for every $(z_n)_{n\in\mathbb{Z}_{>0}} \subseteq \mathbb{C}$ such that $\sum_{n=1}^{\infty} |z_n| < \infty$.

Applications (Muić; Ž.)

A Cuspidal automorphic forms on (the metaplectic cover of) $\mathrm{SL}_2(\mathbb{R})$ and cusp forms of (half-)integral weight:

- I Classical Poincaré series $\psi_{\Gamma,n,k,\chi} \in S_k(\Gamma, \chi)$
- II π being an integrable discrete series of (the metaplectic cover of) $\mathrm{SL}_2(\mathbb{R})$, Poincaré series of K-finite matrix coefficients of π that transform on both sides as characters of K
- III Cusp forms $f_s \in S_k(\Gamma, \chi)$ such that

$$L(s, f) = \langle f,\ f_s \rangle_{S_k(\Gamma,\chi)}\,, \qquad f \in S_k(\Gamma, \chi).$$

B Cuspidal vector-valued modular forms:

- I Classical and elliptic vector-valued Poincaré series.

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Non-vanishing criterion for Poincaré series on the metaplectic cover of $\mathrm{SL}_2(\mathbb{R})$

$$\rightsquigarrow$$

Non-vanishing criterion for Poincaré series of half-integral weight on $\mathcal{H}$

The metaplectic cover of $\mathrm{SL}_2(\mathbb{R})$

Writing $\mathcal{H} := \mathbb{C}_{\Im(z)>0}$,

$$\mathrm{SL}_2(\mathbb{R})^{\sim} := \left\{ \sigma = \left(g_\sigma = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}, \eta_\sigma \right) \in \mathrm{SL}_2(\mathbb{R}) \times \mathbb{C}^{\mathcal{H}} : \right.$$

$$\left. \eta_\sigma \text{ is holomorphic and } \eta_\sigma^2(z) = c_\sigma z + d_\sigma \text{ for all } z \in \mathcal{H} \right\}.$$

Multiplication rule:

$$\sigma_1\sigma_2 := \left(g_{\sigma_1} g_{\sigma_2}, \eta_{\sigma_1}(g_{\sigma_2}.z)\eta_{\sigma_2}(z)\right), \qquad \sigma_1, \sigma_2 \in \mathrm{SL}_2(\mathbb{R})^{\sim}.$$

Left action on $\mathcal{H}$:

$$\sigma.z := \frac{a_\sigma z + b_\sigma}{c_\sigma z + d_\sigma}.$$

For every $k \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$, right action on $\mathbb{C}^{\mathcal{H}}$:

$$\left(f\big|_k \sigma\right)(z) := f\left(\sigma.z\right)\,\eta_\sigma(z)^{-2k}, \qquad z \in \mathcal{H}.$$

The metaplectic cover of $\mathrm{SL}_2(\mathbb{R})$

A smooth covering homomorphism of degree 2:

$$P : \mathrm{SL}_2(\mathbb{R})^{\sim} \to \mathrm{SL}_2(\mathbb{R}), \qquad P(\sigma) := g_\sigma.$$

Using shorthand notation $(g_\sigma, \eta_\sigma(i))$ for $\sigma = (g_\sigma, \eta_\sigma) \in \mathrm{SL}_2(\mathbb{R})^{\sim}$, we have the **Iwasawa parametrization** $\mathbb{R} \times \mathbb{R}_{>0} \times \mathbb{R} \to \mathrm{SL}_2(\mathbb{R})^{\sim}$,

$$(x, y, t) \mapsto \underbrace{\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, 1\right)}_{=:\, n_x \in N} \underbrace{\left(\begin{pmatrix} y^{\frac{1}{2}} & \\ & y^{-\frac{1}{2}} \end{pmatrix}, y^{-\frac{1}{4}}\right)}_{=:\, a_y \in A} \underbrace{\left(\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, e^{i\frac{t}{2}}\right)}_{=:\, \kappa_t \in K}.$$

Haar measure on $\mathrm{SL}_2(\mathbb{R})^{\sim}$: for $\varphi \in C_c\left(\mathrm{SL}_2(\mathbb{R})^{\sim}\right)$,

$$\int_{\mathrm{SL}_2(\mathbb{R})^{\sim}} \varphi(g)\, dg := \frac{1}{4\pi} \int_0^{4\pi} \int_{\mathcal{H}} \varphi(n_x a_y \kappa_t)\, dv(x + iy)\, dt,$$

where $dv(x + iy) := \frac{dx\, dy}{y^2}$ for $x \in \mathbb{R}$ and $y \in \mathbb{R}_{>0}$.

K is a maximal compact subgroup; $\widehat{K} = \left\{ \chi_k(\kappa_t) := e^{-ikt} : k \in \frac{1}{2}\mathbb{Z} \right\}$.

Spaces $S_k(\Gamma, \chi)$ of cusp forms of half-integral weight

From now on, let:

- Γ be a discrete subgroup of finite covolume in $\mathrm{SL}_2(\mathbb{R})^\sim$
- $\chi : \Gamma \to \mathbb{C}^\times$ be a character of finite order
- $k \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$.

A **cusp form** $f \in S_k(\Gamma, \chi)$ is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ such that:

- $f\big|_k \gamma = \chi(\gamma) f$ for all $\gamma \in \Gamma$
- f vanishes at all cusps of $P(\Gamma) := \{g_\gamma : \gamma \in \Gamma\}$.

Petersson inner product on $S_k(\Gamma, \chi)$:

$$\langle f_1, \, f_2 \rangle_{S_k(\Gamma,\chi)} := \frac{1}{\varepsilon_\Gamma} \int_{\Gamma \backslash \mathcal{H}} f_1(z) \overline{f_2(z)} \, \Im(z)^k \, dv(z),$$

where $\varepsilon_\Gamma := \left| \Gamma \cap Z\left(\mathrm{SL}_2(\mathbb{R})^\sim\right) \right|$.

The classical lift $\mathbb{C}^{\mathcal{H}} \to \mathbb{C}^{\mathrm{SL}_2(\mathbb{R})^\sim}$

... is defined by

$$f : \mathcal{H} \to \mathbb{C} \;\mapsto\; F_f : \mathrm{SL}_2(\mathbb{R})^\sim \to \mathbb{C},$$
$$F_f(\sigma) := \left(f\big|_k \sigma\right)(i),$$

restricts to an isometry

$$S_k(\Gamma, 1) \to \mathcal{A}_{cusp}\left(\Gamma \backslash \mathrm{SL}_2(\mathbb{R})^\sim\right) \subseteq L^2\left(\Gamma \backslash \mathrm{SL}_2(\mathbb{R})^\sim\right),$$

and maps

$$P_{\Lambda\backslash\Gamma,\chi} f := \sum_{\gamma \in \Lambda\backslash\Gamma} \overline{\chi(\gamma)}\, f\big|_k \gamma \;\mapsto\; P_{\Lambda\backslash\Gamma,\chi} F_f = \sum_{\gamma \in \Lambda\backslash\Gamma} \overline{\chi(\gamma)}\, F_f(\gamma\,\cdot\,).$$

Non-vanishing criterion for Poincaré series on $\mathcal{H}$

Let $f : \mathcal{H} \to \mathbb{C}$ be a measurable function such that:

- $f\big|_k \lambda = \chi(\lambda) f, \quad \lambda \in \Lambda$
- $\int_{\Lambda\backslash\mathcal{H}} \left| f(z) \Im(z)^{\frac{k}{2}} \right| \, dv(z) < \infty.$

Then,

$$\int_{\Gamma\backslash\mathcal{H}} \left| \left(P_{\Lambda\backslash\Gamma,\chi} f\right)(z) \Im(z)^{\frac{k}{2}} \right| \, dv(z) < \infty.$$

Theorem 2

(1) *If* $\chi\big|_{\Gamma\cap Z(\mathrm{SL}_2(\mathbb{R})^\sim)} \neq \chi_k\big|_{\Gamma\cap Z(\mathrm{SL}_2(\mathbb{R})^\sim)}$, *then* $P_{\Lambda\backslash\Gamma,\chi} f \equiv 0$.

(2) *If* $\chi\big|_{\Gamma\cap Z(\mathrm{SL}_2(\mathbb{R})^\sim)} = \chi_k\big|_{\Gamma\cap Z(\mathrm{SL}_2(\mathbb{R})^\sim)}$, *then* $P_{\Lambda\backslash\Gamma,\chi} f \not\equiv 0$ *if there exists a Borel-measurable set* $S \subseteq \mathcal{H}$ *such that:*

  (1) $\forall z_1, z_2 \in S \quad z_1 \neq z_2 \;\Rightarrow\; \Gamma.z_1 \neq \Gamma.z_2$.

  (2) $\displaystyle\int_{\Lambda\backslash\Lambda.S} \left| f(z)\Im(z)^{\frac{k}{2}} \right| dv(z) > \frac{1}{2} \int_{\Lambda\backslash\mathcal{H}} \left| f(z)\Im(z)^{\frac{k}{2}} \right| dv(z)$ *for some measurable function* $|\cdot| : \mathbb{C} \to \mathbb{R}_{\geq 0}$ *satisfying* (B1) – (B3).

L-functions of cusp forms of half-integral weight

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Let:

- $k \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$
- Γ be a discrete subgroup of finite covolume in $\mathrm{SL}_2(\mathbb{R})^\sim$ such that ∞ is a cusp of $P(\Gamma)$
- $\chi : \Gamma \to \mathbb{C}^\times$ be a character of finite order such that

$$\chi(\gamma) = \eta_\gamma^{-2k}, \qquad \gamma \in \Gamma_\infty,$$

- $h \in \mathbb{R}_{>0}$ such that $Z\left(\mathrm{SL}_2(\mathbb{R})^\sim\right)\Gamma_\infty = Z\left(\mathrm{SL}_2(\mathbb{R})^\sim\right)\langle n_h \rangle$.

The L-**function** of a cusp form $f(z) = \sum_{n=1}^\infty a_n(f) e^{2\pi i n \frac{z}{h}}$ in $S_k(\Gamma, \chi)$ is the function $L(\,\cdot\,, f) : \mathbb{C}_{\Re(s) > \frac{k}{2}+1} \to \mathbb{C}$,

$$L(s, f) := \sum_{n=1}^\infty \frac{a_n(f)}{n^s}.$$

Theorem 3 (Analytic continuation of L-functions)

Suppose $k \in \frac{9}{2} + \mathbb{Z}_{\geq 0}$. Let $f \in S_k(\Gamma, \chi)$. Then, for $\Re(s) < \frac{k}{2}$ the series

$$\Psi_{\Gamma,k,\chi,s} := P_{\Gamma_\infty \backslash \Gamma,\chi} \left(\sum_{n=1}^{\infty} n^{s-1} e^{2\pi i n \frac{\cdot}{h}} \right)$$

converges absolutely and uniformly on compact sets in $\mathcal{H}$ and defines an element of $S_k(\Gamma, \chi)$, and the formula

$$L(s, f) = \frac{\varepsilon_\Gamma (4\pi)^{k-1}}{h^k \Gamma(k-1)} \left\langle f,\ \Psi_{\Gamma,k,\chi,k-\bar{s}} \right\rangle_{S_k(\Gamma,\chi)}$$

defines a holomorphic continuation of $L(\,\cdot\,, f)$ to the half-plane $\mathbb{C}_{\Re(s) > \frac{k}{2}}$.

Theorem 4 (Non-vanishing of L-functions)

Suppose that $k \in \frac{9}{2} + \mathbb{Z}_{\geq 0}$. Let $\frac{k}{2} < \Re(s) < k - 1$. Let us denote

$$N := \inf \left\{ |c| \neq 0 : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in P(\Gamma) \right\} > 0.$$

If $\dfrac{Nh}{\pi}$ is greater than or equal to

$$\max \left\{ \frac{4}{k - \frac{8}{3}}, \left(\frac{e^{\frac{\pi}{2}|\Im(s)|} \Gamma\left(\frac{k - \Re(s) + 1}{2}\right) \Gamma\left(\frac{k - \Re(s) - 1}{2}\right) 2^{\frac{k}{2} - 1}}{\pi \Gamma\left(\frac{k}{2} - 1\right) \left(\Re(s) - \frac{k}{2}\right)} \right)^{\frac{1}{\Re(s) - \frac{k}{2}}} \right\},$$

then

$$L\left(s, \Psi_{\Gamma, k, \chi, k - \overline{s}}\right) > 0.$$

Proof.

Put $S :=]0, h] \times \left] \frac{1}{N}, \infty \right[$ and $| \cdot | := | \cdot |$ in Theorem 2. □

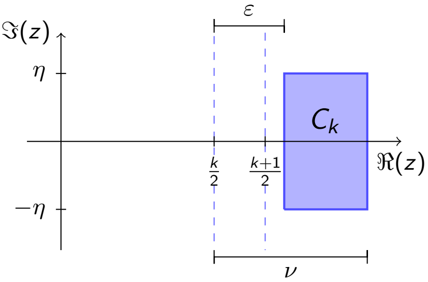
Corollary

Let $\eta, \varepsilon, \nu \in \mathbb{R}_{>0}$ such that

$$\frac{1}{2} < \varepsilon < \nu.$$

For $k \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$ we define

$$C_k := \left[\frac{k}{2} + \varepsilon, \frac{k}{2} + \nu\right] \times [-\eta, \eta] \subseteq \mathbb{C}.$$



There exists $k_0 \in \frac{9}{2} + \mathbb{Z}_{\geq 0}$ such that for every choice of

- $k \in k_0 + \mathbb{Z}_{\geq 0}$
- $s \in C_k$
- a discrete subgroup Γ of finite covolume in $\mathrm{SL}_2(\mathbb{R})^\sim$ such that ∞ is a cusp of $P(\Gamma)$
- a character $\chi : \Gamma \to \mathbb{C}^\times$ of finite order satisfying $\chi(\gamma) = \eta_\gamma^{-2k}$ for all $\gamma \in \Gamma_\infty$

we have

$$L\left(s, \Psi_{\Gamma,k,\chi,k-\bar{s}}\right) > 0.$$

A non-vanishing criterion for vector-valued Poincaré series on $\mathcal{H}$

Basics

$\mathrm{SL}_2(\mathbb{R})$ acts on $\mathcal{H}$ by Möbius transformations:

$$g.\tau = \frac{a\tau + b}{c\tau + d}, \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}),\ \tau \in \mathcal{H}.$$

Let:

- v be the standard $\mathrm{SL}_2(\mathbb{R})$-invariant Radon measure on $\mathcal{H}$:

$$d\mathrm{v}(x + iy) = \frac{dx\, dy}{y^2}$$

- $v : \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{C}_{|z|=1}$ be a multiplier system of weight $k \in \mathbb{R}$.

Poincaré series on $\mathcal{H}$

Let:

- $\Lambda \subseteq \Gamma$ be subgroups of $\mathrm{SL}_2(\mathbb{Z})$ such that $|\mathrm{SL}_2(\mathbb{Z}) : \Gamma| < \infty$
- $\rho : \Gamma \to \mathrm{GL}_n(\mathbb{C})$ be a unitary representation.

Γ acts on the right on $(\mathbb{C}^n)^{\mathcal{H}}$:

$$\left(f\big|_{k,\rho}\gamma\right)(\tau) = v(\gamma)^{-1} j(\gamma,\tau)^{-k} \rho(\gamma)^{-1} f(\gamma.\tau), \quad \tau \in \mathcal{H}.$$

For every measurable function $f : \mathcal{H} \to \mathbb{C}^n$ such that

$$f\big|_{k,\rho}\lambda = f, \qquad \lambda \in \Lambda,$$

we define the **Poincaré series**

$$P_{\Lambda\backslash\Gamma,\rho} f := \sum_{\gamma \in \Lambda\backslash\Gamma} f\big|_{k,\rho}\gamma.$$

It converges absolutely a.e. on $\mathcal{H}$ if $\int_{\Lambda\backslash\mathcal{H}} \|f(\tau)\| \, \Im(\tau)^{\frac{k}{2}} dv(\tau) < \infty$.

Theorem 5 (Integral non-vanishing criterion)

Suppose that $-I_2 \in \Lambda$. Let f be such that the series $P_{\Lambda\backslash\Gamma,\rho} f$ converges absolutely a.e. on $\mathcal{H}$. Then,

$$\int_{\Gamma\backslash\mathcal{H}} \left\| \left(P_{\Lambda\backslash\Gamma,\rho} f\right)(\tau) \right\| \Im(\tau)^{\frac{k}{2}}\, d\mathrm{v}(\tau) > 0$$

if there exists a Borel-measurable set $A \subseteq \mathcal{H}$ with the following properties:

(A1) No two points of A are mutually Γ-equivalent.

(A2) Denoting $(\Lambda.A)^c := \mathcal{H} \setminus \Lambda.A$, we have

$$\int_{\Lambda\backslash\Lambda.A} \|f(\tau)\| \Im(\tau)^{\frac{k}{2}}\, d\mathrm{v}(\tau) > \int_{\Lambda\backslash(\Lambda.A)^c} \|f(\tau)\| \Im(\tau)^{\frac{k}{2}}\, d\mathrm{v}(\tau).$$

An example application

We proved the non-vanishing of the **classical vector-valued Poincaré series**

$$\Psi_{\Gamma,\rho,k,\nu,u} := P_{\Gamma_\infty\backslash\Gamma,\rho}\left(e^{2\pi i\nu\tau}\, u\right)$$

for $k > \frac{8}{3}$, $\Gamma \in \{\Gamma_0(N), \Gamma_1(N), \Gamma(N)\}$ and some suitable choices of:

- a unitary representation $\rho : \Gamma \to \mathrm{GL}_n(\mathbb{C})$
- $\nu \in \mathbb{Q}_{>0}$ such that $\nu \leq \frac{N}{4\pi}\left(k - \frac{8}{3}\right)$
- $u \in \mathbb{C}^n \setminus \{0\}$

by applying Theorem 5 with

$$A =]0, M] \times \left]\frac{1}{N}, \infty\right[\subseteq \mathcal{H},$$

where $M = \begin{cases} 1, & \text{if } \Gamma \in \{\Gamma_0(N), \Gamma_1(N)\} \\ N, & \text{if } \Gamma = \Gamma(N). \end{cases}$

Thank you!